Variational Solution of Blasius Flow for Skin Friction and Heat Transfer

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Abstract

THE objective of the present work is to apply recent developments in thermodynamics of irreversible processes in order to obtain rapid analytical solutions of skin friction and heat transfer in Blasius flow by a variational technique. According to the theory of boundary-layer flows, the irreversible processes of momentum and heat transfer in flows around bodies occur mainly inside a thin layer adjacent to the surface of the body. Therefore, the natural way of studying these nonequilibrium processes is by using the technique based on irreversible thermodynamics. A variational principle introduced by Gyarmati[1] into nonequilibrium thermodynamics is applied to achieve an analytical solution that is unique and the first of its kind because it forms a complete solution to the problem.

Contents

Formulation of Gyarmati's Principle

Gyarmati[1,2] developed a variational principle

$$\delta\int_V (\sigma-\phi-\psi)\mathrm{d}V=0 \tag{1}$$

which describes the evolution of linear, quasilinear, and nonlinear dissipative processes in space and time. When the governing equations of the present system are given in the following balance form:

$$\nabla\cdot V=0,\ \ (V=iu+jv)$$

$$\rho(V\cdot\nabla)V+\nabla\cdot\bar{\bar{P}}=0$$

$$\rho C_p(V\cdot\nabla)T+\nabla\cdot Jq=0 \tag{2}$$

the principle [Eq. (1)] assumes the following:

$$\delta\int_0^\ell\int_0^\infty\left[-Jq\frac{\partial \ell n T}{\partial y}-P_{12}\frac{\partial u}{\partial y}-\frac{L_{\lambda\lambda}}{2}\left(\frac{\partial \ell n T}{\partial y}\right)^2\right.$$

$$\left.-\frac{L_{\lambda\lambda}}{2}\left(\frac{\partial u}{\partial y}\right)^2\frac{Jq^2}{2L_{\lambda\lambda}}-\frac{P_{12}^2}{2L_{\lambda\lambda}}\right]\mathrm{d}y\mathrm{d}x=0 \tag{3}$$

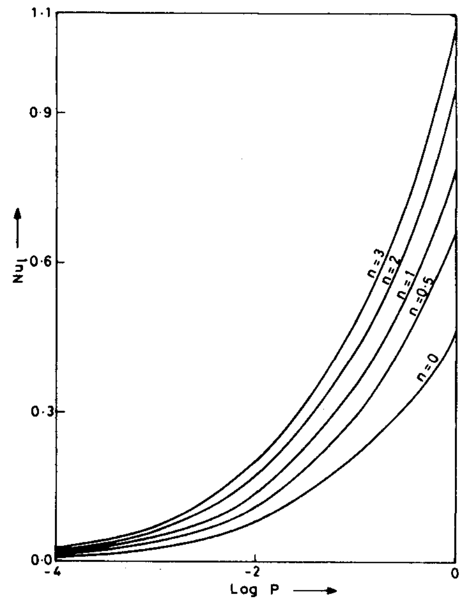

Fig. 1 Variation of local Nusselt number with log$P$ ($P=10^{-4}$ to 1) for various n.

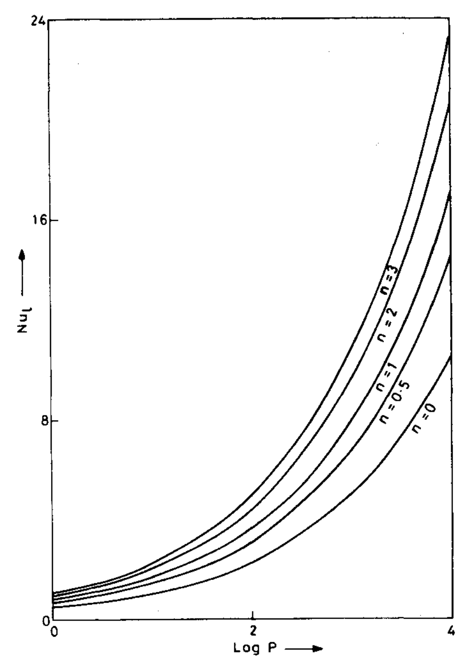

Fig. 2 Variation of local Nusselt number with log$P$ ($P=1$ to 10^4) for various n.

Received Sept. 8, 1981; synoptic received Jan. 7, 1986. Full paper available from National Technical Information Service, Springfield, VA 22151 at the standard price (available upon request).

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Method of Solution

The velocity and temperature distributions inside their respective boundary layers are assumed to be as follows:

$$u/U_\infty = 2y/d_1 - 2y^3/d_1^3 + y^4/d_1^4, \quad (y<d_1); \quad u=U_\infty\,(y\geq d_1)$$

$$(T-T_\infty)/(T_0-T_\infty) = 1-2y/d_2+2y^3/d_2^3-y^4/d_2^4, \quad (y<d_2); \quad T=T_\infty\,(y\geq d_2) \tag{4}$$

The variational principle is formulated for the cases $P\leq 1$ and $P\geq 1$, respectively. The transformations

$$d_{1,2} = d^*_{1,2}\sqrt{\nu x/U_\infty} \tag{5}$$

render the Euler-Lagrange equations of Gyarmati's principle (3) into the following algebraic polynomial equations:

$$d_1^{*4} - 980.0253 = 0 \text{ or } d_1^* = 5.595119 \tag{6}$$

$$\begin{aligned}
&d_2^{*11}P^2[17.12121n^2+25.68182n+11.1039] - d_2^{*9}[P\{117.4603n+104.3651\}+P^2d_1^{*2}\{20n^2+10n\}] - d_2^{*7}[742.857\\
&+Pd_1^2n(133.333)-P^2d_1^{*4}\{21.255n^2-64.20635n\}] + d_2^{*6}[Pd_1^{*3}\{190.4763n-552.3809\}-P^2d_1^{*5}\{5.2473n^2+56.88005n\\
&+115.1546\}] + d_2^{*5}[Pd_1^{*4}n(64.287)-P^2d_1^{*6}\{17.3127n^2-430.9593n-53.42856\}] - d_2^{*4}[Pd_1^{*5}\{200n-2355.556\}\\
&-P^2d_1^{*7}\{15.9476n^2-523.9081n+273.6516\}] + d_2^{*3}[Pd_1^{*6}\{63.492n-1484.127\}-P^2d_1^{*8}\{3.7495n^2\times 188.8141n\\
&+180.2096\}] + d_2^{*2}[Pd_1^{*7}\{46.7532n-1788.313\}-P^2d_1^{*9}\{1.60965n^2-87.23386n+267.3784\}] - d_2^*[Pd_1^{*8}\{35.3535n\\
&-2091.163\}-P^2d_1^{*10}\{1.07506n^2-88.26085n+322.2646\}] + Pd_1^{*9}\{6.8376n-598.2904\}\\
&-P^2d_1^{*11}\{0.175064n^2-19.82244n+94.47332\}=0, \qquad (P\leq 1)
\end{aligned} \tag{7}$$

and

$$\begin{aligned}
&d_2^{*12}P^2[0.8198498n^2+0.9018346n+0.2564931] - d_2^{*11}d_1^*P^2[5.368025n^2+5.981943n+1.728144] + d_2^{*10}d_2^{*2}P^2[8.760358n^2\\
&+9.855411n+2.890951] + d_2^{*9}d_1^{*3}P^2[19.58782n^2+23.7555n+7.374854] - d_2^{*8}d_1^{*4}P^2[63.41435n^2+76.80094n+24.19804]\\
&-d_2^{*6}[Pd_1^{*4}\{233.9327n+127.9276\}-P^2d_1^{*6}\{112.6651n^2+140.8313n+47.73006\}] + d_2^{*5}d_1^{*5}P[678.2108n+383.8384]\\
&-d_2^{*3}d_1^{*7}P[1587.302n+1063.492] - 7428.57d_1^{*8}=0, \qquad (P\geq 1)
\end{aligned} \tag{8}$$

The polynomial equations (6-8) offer any practicing engineer a rapid way of obtaining solutions for arbitrary values of n and P.

Analysis of Results

The present approximate method produces results more exact than those of Karman-Pohlhausen's momentum integral method. For the skin friction of our procedure, 0.4737 is closer to the exact value 0.4696 (Ref. 3) than Karman-Pohlhausen's value of 0.4851. In Table 1, the present results are compared with known numerical and approximate solutions.[4]

Table 1 Comparison of numerical and approximate values of local Nusselt number ($P=0.72$)

	Numerical values		Approximate Values			
n	Chapman-Rubesin	Levy	Lighthill-Fettis	Schuh	Imai	Present values
0	0.296	0.2901	0.304	0.0	0.3011	0.2972
1	0.489	0.4814	0.490	0.4524	0.4845	0.4907
2	0.597	0.5865	0.594	0.5700	0.5903	0.5970
3	0.684	0.6634	0.671	0.6525	0.6680	0.6752
4	0.744	0.7268	0.734	0.7182	0.7310	0.7388

Since Levy has estimated the accuracy of his results to be within 4%, one can consider the present results to be completely satisfactory.

Figures 1 and 2 demonstrate that the heat transfer increases with n. It also follows that the increase in Nusselt number is quite rapid in the case of larger n. Further, when the Prandtl number tends to zero, the heat transfer approaches a limiting value.

Acknowledgment

The authors are grateful to the reviewer for his valuable suggestions.

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